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Diffraction of shock waves by a moving thin wing

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An analytical solution is obtained for the flow field due to the impinging of a plane shock wave of arbitrary strength by a thin wing moving in the opposite direction. The planform and the thickness distribution of the wing can be arbitrary and the speed of the wing can be either supersonic or subsonic relative to the undisturbed stream ahead of the shock or to that behind the shock. The solution is a generalization of the previous solution of Ting & Ludloff for the diffraction of shock wave by a two-dimensional stationary airfoil to a three-dimensional wing moving with supersonic or subsonic speed relative to the stream ahead of or behind the shock. The solution is employed for the analysis of the changes in aerodynamic forces when an airplane encounters a blast wave or a shock wave of another airplane. It is also used to study the diffraction of a shock wave or an N-wave advancing over flat terrains.

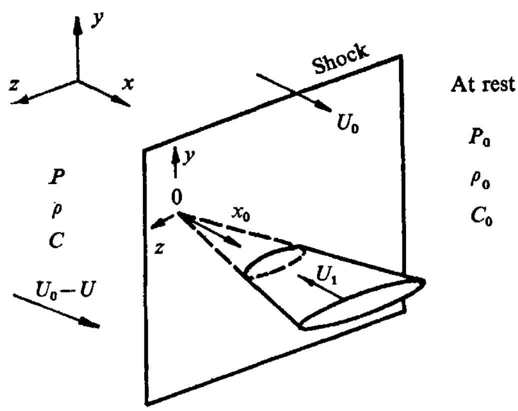
1. Introduction

The variations in aerodynamic forces on an airplane, when it encounters a shock wave due to an explosion or that of another vehicle nearby, are of practical interest (figure 1 (*a*)). The problem of the diffraction of a shock wave or an N-wave advancing over a flat terrain is an area of interest in the current sonic boom investigations (figure 1 (*b*)). The second problem can be considered as a special case of the first one, i.e. the diffraction of a shock wave advancing over a stationary symmetric thin wing. In this paper, analytical solutions for both problems are presented.

The solution for the conical flow field due to the diffraction of a shock wave advancing over a stationary thin wedge was obtained by Lighthill (1949). Extension to stationary wedges at yaw and to wedges moving head on with supersonic speed were obtained by Chester (1954) and by Smyrl (1963), respectively. Additional conical solutions have been developed by Blankenship (1965) for the diffraction of a shock wave by a slender cone moving with supersonic speed and by Ter-Minassiants (1969) for the diffraction of an oblique shock wave and its regular reflected wave by a small corner.

The solution for the diffraction of a shock wave by any stationary two-dimensional symmetric thin airfoil was obtained by Ting & Ludloff (1952) directly as a solution of the two-dimensional wave equation. The same method was applied to a stationary slender axially symmetric body by Ludloff & Friedman (1952). The boundary condition for the disturbance pressure p' on the airfoil or the body behind the shock was fulfilled by an appropriate source distribution.

The homogeneous boundary condition across the shock, $D_{xt}p' = 0$, is replaced by an equivalent boundary condition or a fictitious source distribution on the plane of the wing or along the axis of the body ahead of the shock. The fictitious source distribution is related to the given source distribution behind the shock



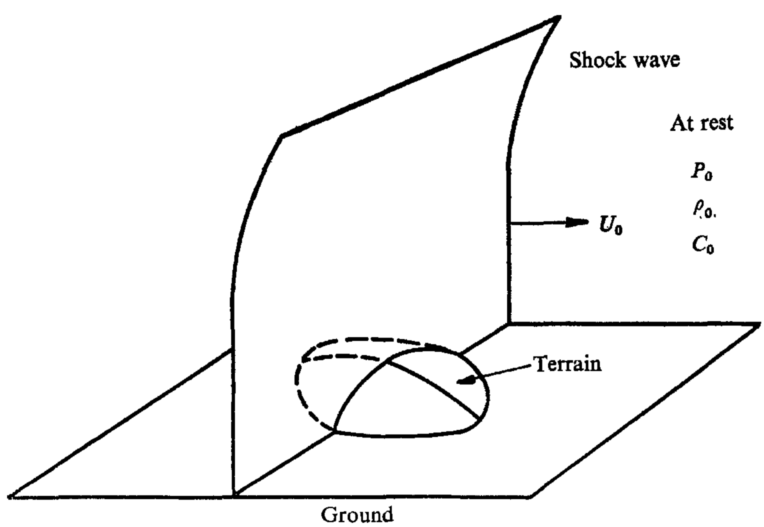


FIGURE 1. Diffraction problems: (*a*) diffraction of shock by a moving wing, (*b*) shock wave advancing over a flat terrain.

by a linear transform of the independent variables. The final solution is given by the known integral solution for unsteady linear source distributions.

Following the formulation of Ting & Ludloff (1952), Arora obtained analytic solutions for the diffraction of a shock wave by a slender body (1968) and by a planar symmetric thin wing (1969). The solutions of Arora, which were obtained by a different procedure using Laplace and Fourier transforms, can again be recognized as integral solutions of unsteady axial or planar source distributions.

In the present paper, the procedure used by Ting & Ludloff (1952) is extended to the unsteady three-dimensional problem, i.e. the diffraction of a shock wave by a moving planar symmetric thin wing. The governing differential equations and the boundary conditions are formulated in § 2. The shock condition is now of the form $D_{xt}p' = KC^2p'_{0,x_0x_0}$, where C is the speed of sound behind the shock. The inhomogeneous term is due to the disturbed pressure p'_0 created ahead of the shock by the moving wing. The pressure p'_0 is given by a steady flow solution without the shock and with the speed of sound the same as that of the stream ahead of the shock, C_0. In the appendix, a steady solution for an equivalent wing moving in a stream without a shock and with the speed of sound C can be found so that the corresponding disturbance pressure p^* creates the same inhomogeneous term in the shock condition, i.e. $D_{xt}p^* = KC^2p'_{0,x_0x_0}$. The difference between the disturbance pressure behind the shock, p', and p^* obeys the homogeneous shock condition and is obtained in § 3 by an extension of the procedure of Ting & Ludloff (1952). The complete analytic solution is given in § 4 together with a list of the relevant symbols. A physical interpretation of the individual terms in the solution as integral solutions for moving planar source distributions is also presented. In § 5 the analytical solution is reduced to a sum of 'quasi-steady' three-dimensional solutions, so that it is easier to carry out the integrations for a given wing. Furthermore, from the planform of the wing, a domain of influence of the shock can be defined, and outside that domain the analytic solution can be reduced to the sum of at most two steady three-dimensional solutions. In § 6, the integrals are evaluated for a simple semi-infinite swept back wing so that explicit solutions are presented. By the superposition of these explicit solutions, solutions for wings with complicated planforms and thickness distributions can be obtained in the same manner as in the steady flow problems (Donovan & Lawrence 1957). Several numerical examples are included, e.g. the diffraction of a shock wave by a flat terrain in the shape of a pyramid and the variation of the lift and drag of a triangular wing with supersonic edges impinging on a shock wave.

It should be pointed out here that the solutions of Smyrl (1963) and Arora (1968, 1969) are restricted to bodies or wings moving at supersonic speed relative to the flow ahead of the shock. The analytic solution presented in this paper is valid regardless of whether the wing is moving at supersonic or subsonic speed relative to the stream ahead of or behind the shock.

2. Formulation of the problem

Figure 1 (*a*) shows a thin wing lying in the x–z plane and impinging head on to a plane shock wave moving in the direction of the x-axis. The undisturbed flow ahead of the shock is at rest with pressure P_0, density ρ_0 and speed of sound C_0 or $(\gamma P_0/\rho_0)^{\frac{1}{2}}$. The shock front is advancing with velocity U_0 and the undisturbed uniform stream behind the shock is moving with velocity $U_0 - U$, pressure P and density ρ and speed of sound C or $(\gamma P/\rho)^{\frac{1}{2}}$. Relative to the shock front the Mach number ahead of the shock, $M_0 = U_0/C_0$ and that behind the shock, $M = U/C$, are related by the normal shock condition (Liepman & Roshko 1957) with

$M_0 > 1$ and $M < 1$. Likewise the pressure ratio, density ratio and the ratio of speed of sounds are related to M_0 or M (Liepman & Roshko 1957).

The thin wing is moving in the direction of the negative x-axis with velocity U_1 relative to the undisturbed stream ahead of the shock. The velocity U_1 can be supersonic or subsonic relative to C_0. The velocity of the wing relative to the undisturbed stream behind the shock is U_1+U_0-U, which can be either supersonic or subsonic relative to C.

For a symmetric wing at zero angle of attack, the disturbed flow is symmetric with respect to the variable y. It suffices to consider only $y \geqslant 0$. With ϵ as the small thickness parameter, the linearized disturbance pressure, density and velocity components behind the shock will be denoted by $\epsilon p'$, $\epsilon\rho'$, $\epsilon u'$, $\epsilon v'$ and $\epsilon w'$ respectively. For the regions ahead of the shock, these disturbance quantities will be represented by the same symbols with subscript 0, namely, $\epsilon p'_0$, $\epsilon\rho'_0$, $\epsilon u'_0$, $\epsilon v'_0$ and $\epsilon w'_0$.

The linearized boundary condition on the plane of the wing is

$$\epsilon v' = \epsilon(U_1+U_0-U)f_{x_0}(x_0,z) \quad \text{for} \quad x_0 < (U_1+U_0)t, \tag{2.1}$$

and

$$\epsilon v'_0 = \epsilon U_1 f_{x_0}(x_0,z) \quad \text{for} \quad x_0 > (U_1+U_0)t, \tag{2.2}$$

where x_0 is fixed on the wing surface and $y = f(x_0,z)$ represents the upper surface of the wing inside the planform S. Outside the planform S, $f(x_0,z)$ vanishes.

Since the shock front is moving with supersonic speed ($M_0 > 1$) relative to the undisturbed stream ahead of the shock, the presence of the shock will not influence the flow field ahead of it. The flow field ahead of the shock is therefore a steady isentropic flow in variables x_0, y, z. The governing equations are,

$$\left.\begin{aligned} \rho_0 U_1 u'_0 = -p'_0, \quad \rho_0 U_1 (v'_0)_{x_0} = -(p'_0)_y,\\ \rho_0 U_1 (w'_0)_{x_0} = -(p'_0)_z, \quad C_0^2\rho'_0 = p'_0,\\ (M_1^2-1)(p'_0)_{x_0x_0} - (p'_0)_{yy} - (p'_0)_{zz} = 0. \end{aligned}\right\} \tag{2.3}$$

The boundary condition (2.2) yields a condition for $(p'_0)_y$ and the solution for the region ahead of the shock is given by the integral,

$$p'_0(x_0,y,z) = \frac{\rho_0 U_1^2}{\pi\nu}\iint \frac{f_{x_0x_0}(\xi_0,\zeta)\,d\xi_0\,d\zeta}{\{(x_0-\xi_0)^2-(M_1^2-1)[(z-\zeta)^2+y^2]\}^{\frac{1}{2}}}. \tag{2.4}$$

For the subsonic case $M_1 = U_1/C < 1$, the domain of integration is the planform of the wing and $\nu = 2$. For the supersonic case $M_1 > 1$, the domain of integration is the part of the planform of the wing inside the hyperbola,

$$\xi < x_0 - \{(M_1^2-1)[(z-\zeta)^2+y^2]\}^{\frac{1}{2}}.$$

For the region behind the shock, the flow field cannot be reduced to a steady flow and will be a function of time t and three variables x, y, z. The co-ordinates are fixed on the undisturbed flow behind the shock. The linearized governing equations are,

$$\left.\begin{aligned} \rho'_t + \rho(u'_x+v'_y+w'_z) = 0,\\ \rho u'_t = -p'_x, \quad \rho v'_t = -p'_y, \quad \rho w'_t = -p'_z, \quad p'_t = C^2\rho'_t. \end{aligned}\right\} \tag{2.5}$$

By straightforward elimination, it is found that the disturbance pressure fulfils the simple wave equation

$$\Box p' = \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}-\frac{1}{C^2}\frac{\partial^2}{\partial t^2}\right)p' = 0, \tag{2.6}$$

while the other quantities fulfil the equation $(\partial/\partial t)(\Box g) = 0$, where g stands for u', v', w' or ρ'.

With (2.6) serving as the governing equation for p' the next step is to state the initial conditions for the region behind the shock, i.e. $x < Ut$.

If the wing is moving at subsonic speed ($M_1 < 1$) relative to the stream ahead of the shock, the initial conditions are

$$p'(x < Ut, y, z, t) \to 0, \quad p'_t(x < Ut, y, z, t) \to 0 \quad \text{as} \quad t \to -\infty. \tag{2.7}$$

The boundary condition at infinity is

$$p'(x < Ut, y, z, t) \to 0, \quad \text{as} \quad (x^2+y^2+z^2)^{\frac{1}{2}} \to \infty. \tag{2.8}$$

For a stationary wing ($U_1 = 0$) or a wing moving with supersonic speed ($U_1/C_0 > 1$) the initial conditions and the boundary condition at infinity can be sharpened but it is not necessary to impose these sharpened ones instead of (2.7) and (2.8).

With x_0 related to x by the translation $x_0 = x + (U_1+U_0-U)t$, the boundary conditions (2.1) and (2.5), yield a condition for p'_y

$$p'_y(x < Ut, 0^+, z, t) = -\rho(U_1+U_0-U)^2 f_{x_0x_0}(x_0, z). \tag{2.9}$$

Relative to the undisturbed flow behind the shock, the air in front of the shock, the wing and the shock front are moving with velocity $-(U_0-U)$, $-(U_1+U_0-U)$ and U, respectively. The disturbed shock front can be expressed by the equation,

$$x = Ut + \epsilon\psi(y,z,t) + O(\epsilon^2). \tag{2.10}$$

Under the framework of linearized theory, the unit normal vector $\hat{n}$ and the shock velocity $U_s\hat{n}$ are related to $\psi(y,z,t)$ as follows,

$$\left.\begin{aligned} \hat{n} &= \hat{\imath} - \epsilon\psi_y\hat{\jmath} - \epsilon\psi_z\hat{k}, \\ U_s\hat{n} &= (U+\epsilon\psi_t)\hat{\imath} - \epsilon U\psi_y\hat{\jmath} - \epsilon U\psi_z\hat{k}. \end{aligned}\right\} \tag{2.11}$$

The continuity of tangential components of velocity across the shock yields

$$v'(x = Ut, y, z, t) = v'_0(x_0 = (U_1+U_0)t, y, z) - (U_0-U)\psi_y(y,z,t), \tag{2.12}$$

$$w'(x = Ut, y, z, t) = w'_0(x_0 = (U_1+U_0)t, y, z) - (U_0-U)\psi_z(y,z,t). \tag{2.13}$$

Solution of the continuity, normal momentum and energy equation across the shock for ρ', u' and p' yields

$$\begin{aligned} C^2\rho'(x = Ut, y, z, t) = (1+\Omega_0)\,p'(x = Ut, y, z, t) \\ + \Omega_3 p'_0(x_0 = (U_1+U_0)t, y, z), \end{aligned} \tag{2.14a}$$

$$\begin{aligned} \rho C u'(x = Ut, y, z, t) = \Omega_1 p'(x = Ut, y, z, t) \\ + \rho C u'_0(x_0 = (U_1+U_0)t, y, z) \\ + \Omega_4 p'_0(x_0 = (U_1+U_0)t, y, z), \end{aligned} \tag{2.14b}$$

$$\begin{aligned} \rho(U-U_0)\,\psi_t(y,z,t) = \Omega_2 p'(x = Ut, y, z, t) \\ + \rho(U-U_0)\,u'_0(x_0 = (U_1+U_0)t, y, z) \\ + \Omega_5 p'_0(x_0 = (U_1+U_0)t, y, z), \end{aligned} \tag{2.14c}$$

where $$\Omega_0 = \frac{-(\gamma-1)(M^2-1)^2}{M^2[2+(\gamma-1)M^2]},$$
$$\Omega_1 = [(3\gamma-1)M^2+3-\gamma]/[2+(\gamma-1)M^2],$$
$$\Omega_2 = -(1-M^2)/2M^2,$$
$$\Omega_3 = (\gamma-1)(M_0^2-1)(1-M^2)/M^2[2+(\gamma-1)M^2],$$
$$\Omega_4 = -\{\gamma+1+M_0^2[2(\gamma-1)M^2+3-\gamma]\}/[2M^2(\gamma+1)],$$
$$\Omega_5 = -[\gamma-3-2(\gamma-1)M^2+(\gamma+1)M_0^2(2M^2-1)]/[2M^2(\gamma+1)].$$

By using differential equations (2.3), (2.5), (2.6), the boundary conditions across the shock $x = Ut$ can be reduced to a single condition on p',

$$D_{x,t}p'(x = Ut, y, z, t) = KC^2 p'_{0,x_0x_0}(x_0 = (U_1+U_0)t, y, z), \tag{2.15}$$

where

$$K = -[(\bar{M}_0+\bar{M}_1)^2[\Omega_4-\bar{M}_0/(\bar{M}_1 M)]+(M_1^2-1)[\Omega_5-(M^{-1}+\bar{M}_1^{-1})\bar{M}_0]M],$$

and $D_{x,t}$ is the linear differential operator defined as

$$D_{x,t} = (\Omega_1+M+\Omega_2 M)\frac{\partial^2}{\partial t^2}+(1+M^2+2M\Omega_1)C\frac{\partial^2}{\partial t\,\partial x}+(\Omega_1 M^2+M-M\Omega_2)C^2\frac{\partial^2}{\partial x^2}. \tag{2.16}$$

The differential operator $D_{x,t}$ is identical with that of the two-dimensional problem (Ting & Ludloff 1952). The inhomogeneous term is the contribution due to the disturbances created by the moving airfoil ahead of the shock.

The boundary conditions along the shock and that along the body surface creates a discontinuity in p_y at their intersection, i.e. $x = Ut$, $y = 0$. Along the shock, (2.12) yields

$$(U-U_0)\psi_y(0^+, z, t) = v'(x = Ut^-, 0^+, z, t) - v'_0(x_0 = (U_1+U_0)t^+, 0^+, z).$$

Since the velocity components are continuous on either side of the shock near its intersection with the body, (2.1), (2.2) and the preceding equation give

$$(U-U_0)\psi_{yt}(0^+, z, t) = (U_1+U_0)(U_0-U)f_{x_0x_0}(x_0 = (U_1+U_0)t, z).$$

From (2.14*c*) along the shock, the following is obtained:

$$p'_y(x = Ut, y = 0^+, z, t) = [4(\gamma+1)^{-1}\rho(U_1+U_0)+\rho_0 U_1^2\Omega_5/\Omega_2]f_{x_0x_0}[(U_0+U_1)t, z]. \tag{2.17}$$

On the other hand (2.1) implies

$$p'_y(x = Ut^-, y = 0, z, t) = -\rho(U_1+U_0-U)^2 f_{x_0x_0}[(U_0+U_1)t, z]. \tag{2.18}$$

Equations (2.17) and (2.18) define the discontinuity in p'_y behind the shock at its intersection with the body. It should be pointed out that in the neighbourhood ahead of the shock there is no such discontinuity in $p'_{0,y}$ since the solution p'_0 is not influenced by the presence of the shock.

The moving front, $x = Ut$, suggests the introduction of new variables $\bar{x}, \bar{y}, \bar{z}, \bar{t}$ from the old variables by the Lorentz transformation (Ting & Ludloff 1952).

$$\left.\begin{aligned}\bar{x} &= (x-Ut)/(1-M^2)^{\frac{1}{2}}, & \bar{y} &= y,\\ \bar{t} &= (Ct-Mx)/(1-M^2)^{\frac{1}{2}}, & \bar{z} &= z.\end{aligned}\right\} \tag{2.19}$$

The region behind the shock, $x < Ut$, becomes the region $\bar{x} < 0$. In this region, the wave equation remains of the same type,

$$p'_{\bar{x}\bar{x}}+p'_{\bar{y}\bar{y}}+p'_{\bar{z}\bar{z}}-p'_{\bar{t}\bar{t}} = 0. \tag{2.20}$$

The initial conditions are

$$p' = p'_{\bar{t}} = 0, \quad \text{as} \quad \bar{t}\to-\infty. \tag{2.21}$$

The boundary conditions become

$$p'\to 0, \quad \text{as} \quad \bar{x}^2+\bar{y}^2+\bar{z}^2\to\infty, \tag{2.22}$$

$$p'_{\bar{y}}(\bar{x},0,\bar{z},\bar{t}) = \rho C^2 A_0 f_{x_0x_0}[x_0 = \bar{a}(\bar{t}+\bar{\lambda}_0\bar{x}),\bar{z}], \tag{2.23}$$

$$\bar{D}_{\bar{x},\bar{t}}\, p'(\bar{x}=0,\bar{y}>0,\bar{z},\bar{t}) = \bar{K}p'_{x_0x_0}[x_0 = \bar{a}\bar{t},\bar{y}>0,\bar{z}], \tag{2.24}$$

and

$$p'_{\bar{y}}(\bar{x}=0^-,\bar{y}=0,\bar{z},\bar{t}) = \rho C^2 A_0 f_{x_0x_0}[x_0=\bar{a}\bar{t},\bar{z}], \tag{2.25}$$

$$p'_{\bar{y}}(\bar{x}=0,\bar{y}=0^+,\bar{z},\bar{t}) = \rho C^2 \mu f_{x_0x_0}[x_0=\bar{a}\bar{t},\bar{z}], \tag{2.26}$$

where

$$\bar{a} = (\bar{M}_1+\bar{M}_0)/(1-M^2)^{\frac{1}{2}}, \quad \bar{M}_0 = U_0/C, \tag{2.27}$$

$$\bar{\lambda}_0 = M+(1-M^2)/(\bar{M}_0+\bar{M}_1), \quad \bar{M}_1 = U_1/C, \tag{2.28}$$

$$A_0 = -(\bar{M}_1+\bar{M}_0-M)^2, \tag{2.29}$$

$$\mu = -4M(2\bar{M}_1+\bar{M}_0)/(\gamma+1)+M_1^2 M\Omega_5/(\bar{M}_0\Omega_2), \tag{2.30}$$

$$\bar{D}_{\bar{x}\bar{t}} = \frac{\partial^2}{\partial\bar{x}^2}+2M\frac{\partial^2}{\partial\bar{x}\,\partial\bar{t}}+\frac{1}{M_0^2}\frac{\partial}{\partial\bar{t}^2}, \tag{2.31}$$

$$\bar{K} = -2M(1-M^2)^{-1}\{(\bar{M}_0+\bar{M}_1)^2[\Omega_4-\bar{M}_0/(M\bar{M}_1)] + M(M_1^2-1)[\Omega_5-(M^{-1}+\bar{M}_1^{-1})\bar{M}_0]\}, \tag{2.32}$$

and Ω_2, Ω_4, Ω_5 are defined by the equations following (2.14*c*). Equations (2.20) to (2.26) summarize the mathematical formulation of the problem.

3. The analytic solution

The solution for the wave equation, (2.20) subjected to the initial condition, (2.21), and the boundary condition at infinity, (2.22), can be related to its normal derivative on the plane $\bar{y} = 0$ by the Kirchhoff formula (Baker & Copson 1950),

$$p'(\bar{x},\bar{y},\bar{z},\bar{t}) = -\frac{1}{2\pi}\iint_{-\infty}^{\infty}\frac{p'_{\bar{y}}(\xi,0,\zeta,\bar{t}-\bar{r})}{\bar{r}}\,d\xi\,d\zeta, \tag{3.1}$$

where $\bar{r} = [(\bar{x}-\xi)^2+\bar{y}^2+(\bar{z}-\zeta)^2]^{\frac{1}{2}}$. The region where $p'_y(\bar{x},0,\bar{y},\bar{t})$ is non-zero will in general be bounded.

For the left half of the plane $y = 0$ $(x < 0)$, p'_0 is given by the boundary condition, (2.23). For the right half of the plane $\bar{y} = 0$ $(\bar{x} > 0)$ p'_y is undefined. The next step is to find a differential equation for $p'_y(\bar{x} > 0, 0, \bar{z}, \bar{t})$ such that the solution given by (3.1) fulfils the condition across the shock (2.24) and possesses the proper discontinuity at $\bar{x} = 0$, $\bar{y} = 0$ of (2.25), (2.26). Prior to doing this, the inhomogeneous terms in the shock condition (2.24) will be removed by splitting the pressure disturbance p' into two terms, each of which is a solution of the wave equation (2.20).

$$p' = \bar{p}+p^*; \tag{3.2}$$

p^* is a solution of the type (3.1) yielding the inhomogeneous term in (2.24).

In the inhomogeneous term the pressure $p_0'(x_0, y, z)$ is given by a steady flow solution of (2.4) with speed of sound C_0. If the co-ordinate x_0 is related to x and then to $\bar{x}$ so that they represent the same point for all x_0, the solution p_0' in the new variables will not fulfil the wave equation with the speed of sound C nor the equivalent equation (2.20). Since it is necessary only to reproduce the inhomogeneous term at $x_0 = (U_0+U_1)\,t$ or at $\bar{x} = 0$ with $\bar{t} = Ct(1-M^2)^{\frac{1}{2}}$, a linear transformation $x_0 = \bar{a}(\bar{t}+\bar{\lambda}^*\bar{x})$ can be introduced with $\bar{\lambda}^*$ to be defined and

$$\bar{a} = (\bar{M}_0+\bar{M}_1)/(1-M^2)^{\frac{1}{2}}. \tag{3.3}$$

The pressure p^* is then defined by an integral of the type in (3.1),

$$p^* = -\frac{\rho C^2 A^*}{2\pi}\iint_{-\infty}^{\infty}\frac{d\xi\, d\zeta}{\bar{r}} f_{x_0 x_0}[\bar{a}(\bar{\lambda}^*\xi+\bar{t}-\bar{r}), \bar{\zeta}]. \tag{3.4}$$

The constants A^* and $\bar{\lambda}^*$ are defined by the condition,

$$D_{\bar{x}\bar{t}}\, p^*(\bar{x} = 0, \bar{y}, \bar{z}, \bar{t}) = \bar{K} p'_{0, x_0 x_0}(x_0 = \bar{a}\bar{t}, \bar{y}, \bar{z}). \tag{3.5}$$

In the appendix it is shown that this condition is fulfilled if

$$\bar{\lambda}^* = [1-(M_1^2-1)/\bar{a}^2]^{\frac{1}{2}}, \tag{3.5a}$$

and

$$A^* = -(M/\bar{M}_0)\,\bar{M}_1^2\bar{K}/[\bar{a}^2 H(-\bar{\lambda}^*)]. \tag{3.5b}$$

In the integrand of (3.4), $f[\bar{a}(\bar{\lambda}^*\bar{x}+\bar{t}), \bar{z}]$ represents an equivalent wing moving with velocity $1/\bar{\lambda}^*$.

Since the wing impinges on the shock at $t = 0$, the particular solution p^* given by (3.4) fulfils all the boundary conditions and initial conditions for the region behind the shock for $t < 0$. For a wing moving at supersonic speed ($M_1 > 1$), p^* is identically zero behind the shock for $t < 0$. At subsonic speed ($M_1 < 1$), p^* gives the disturbance pressure behind the shock for $t < 0$. In either case, it is correct to write,

$$p' = p^* \quad \text{and} \quad \bar{p} = 0, \quad \text{for} \quad x < Mt \quad (t < 0). \tag{3.6}$$

After the impingement of the shock by the wing, $\bar{t} > 0$, the solution p^* alone will not fulfil the boundary condition at $\bar{y} = 0$, $\bar{x} < 0$, (2.23). The additional contribution $\bar{p}$ should also fulfil the wave equation, (2.20), the initial condition (2.21) and the condition at infinity (2.22). The remaining boundary conditions for p', (2.23) to (2.26), become respectively,

$$\bar{p}_{\bar{y}}(\bar{x} < 0, 0, \bar{z}, \bar{t}) = \rho C^2\{A_0 f_{x_0 x_0}[\bar{a}(\bar{t}+\bar{\lambda}_0\bar{x}), \bar{z}]+A_5 f_{x_0 x_0}[\bar{a}(\bar{t}+\bar{\lambda}_5\bar{x}), \bar{z}], \tag{3.7}$$

$$\bar{D}_{\bar{x},\bar{t}}\,\bar{p}(\bar{x} = 0, \bar{y} > 0, \bar{z}, \bar{t}) = 0, \tag{3.8}$$

and

$$\bar{p}_{\bar{y}}(\bar{x} = 0^-, \bar{y} = 0, \bar{z}, \bar{t}) = \rho C^2[A_0+A_5] f_{x_0 x_0}(\bar{a}\bar{t}, \bar{z}), \tag{3.9}$$

$$\bar{p}_{\bar{y}}(\bar{x} = 0, \bar{y} = 0^+, \bar{z}, \bar{t}) = \rho C^2[\mu+A_5] f_{x_0 x_0}(\bar{a}\bar{t}, \bar{z}), \tag{3.10}$$

where $A_5 = -A^*$ and $\bar{\lambda}_5 = \bar{\lambda}^*$.

The condition (3.8) across the shock for $\bar{p}$ is homogeneous and the solution for $\bar{p}$ will be obtained in the same manner as that for a two-dimensional stationary wing (Ting & Ludloff 1952). $\bar{p}$ will be expressed in terms of $\bar{p}_{\bar{y}}$ on the plane $y = 0$ by Kirchhoff's formula,

$$\bar{p}(\bar{x} < 0, \bar{y}, \bar{z}, \bar{t}) = -\frac{\rho C^2}{2\pi}\int_{-\infty}^{\infty} d\zeta \int_{-\infty}^{0} d\xi \{A_0 f_{\xi_0\xi_0}[\xi_0, \zeta] + A_5 f_{\xi_5\xi_5}[\xi_5, \zeta]\}/\bar{r}$$
$$-\frac{\rho C^2}{2\pi}\int_{-\infty}^{\infty} d\zeta \int_{0}^{\infty} d\xi \{\bar{p}_{\bar{y}}(\xi, 0, \zeta, \bar{t}-\bar{r})\}\bar{r}, \tag{3.11}$$

where $\xi_i = (\bar{\lambda}_i \xi + \bar{t} - \bar{r})\bar{a} \quad (i = 1, 5).$

The unknown distribution $\bar{p}_{\bar{y}}(\bar{x}, 0, \bar{z}, \bar{t})$ ahead of the shock, $\bar{x} > 0$, will be defined by the remaining boundary conditions (3.8) and (3.10). By observing the identity $[g(\xi, \zeta, \bar{t}-\bar{r})/\bar{r}]_{\bar{x}} = -[g(\xi, \zeta, \bar{t}-\bar{r})/\bar{r}]_{\xi} + [g_{\xi}(\xi, \zeta, \tau)_{\tau=\bar{t}-\bar{r}}]/\bar{r}$, the differential operator $\bar{D}_{\bar{x},\bar{t}}$, is applied to $\bar{p}$,

$$\bar{D}_{\bar{x},\bar{t}}\ \bar{p}(\bar{x} = 0, \bar{y} > 0, \bar{z}, \bar{t}) = -\frac{\rho C^2 \bar{a}^2}{2\pi}\int_{-\infty}^{\infty} d\zeta \int_{-\infty}^{0} d\xi \{A_0 H(-\bar{\lambda}_0) f^{(\mathrm{IV})}(\xi_0, \zeta)$$
$$+ A_5 H(-\bar{\lambda}_5) f^{(\mathrm{IV})}(\xi_5, \zeta)\}/\bar{r} - \frac{1}{2\pi}\int_{-\infty}^{\infty} d\zeta \int_{0}^{\infty} d\xi [\bar{D}_{\xi\tau}\bar{p}_{\bar{y}}(\xi, 0, \zeta, \tau)]_{\tau=\bar{t}-\bar{r}}/\bar{r}$$
$$-\frac{1}{2\pi}\int_{-\infty}^{\infty} d\zeta \{[\Delta_2(\zeta, \bar{t}-\bar{r}) + 2M\Delta_{1,\bar{t}}(\zeta, \bar{t}-\bar{r})]/\bar{r}\}_{\xi=0}, \tag{3.12}$$

where
$$H(\bar{\lambda}) = 1/M_0^2 - 2M\bar{\lambda} + \bar{\lambda}^2,$$
$$\Delta_1(\zeta, \bar{t}) = \bar{p}_{\bar{y}}(0^+, 0, \zeta, \bar{t}) - \bar{p}_{\bar{y}}(0^-, 0, \zeta, \bar{t}),$$
$$\Delta_2(\zeta, \bar{t}) = \bar{p}_{\bar{y}\bar{x}}(0^+, 0, \zeta, \bar{t}) - \bar{p}_{\bar{y}\bar{x}}(0^-, 0, \zeta, \bar{t}),$$

and $f^{(\mathrm{IV})}$ means the fourth derivative of f with respect to its first argument. Boundary condition (3.8) implies that the expression (3.12) vanishes. This will be the case if,

$$\bar{D}_{\bar{x}\bar{t}}\bar{p}_{\bar{y}}^{(1)}(\bar{x} > 0, \bar{z}, \bar{t})$$
$$= -\rho C^2 \bar{a}^2 \{A_0 H(-\bar{\lambda}_0) f^{(\mathrm{IV})}[\bar{a}(\bar{t} - \bar{\lambda}_0 \bar{x}), \bar{z}] + A_5 H(-\bar{\lambda}_0) f^{(\mathrm{IV})}[\bar{a}(\bar{t} - \bar{\lambda}_5 \bar{x}), \bar{z}]\}/2\pi, \tag{3.13}$$

and
$$\Delta_2(\bar{z}, \bar{t}) = -2M\Delta_{1,t}(\bar{z}, \bar{t}). \tag{3.14}$$

For the fulfilment of the boundary condition (3.10), it is necessary to specify the appropriate limit $\bar{p}_{\bar{y}}$ as $\bar{x} \to 0^+$ along the $\bar{x}$–$\bar{z}$ plane. The limit is defined by applying a kind of 'mean value theorem' for $\bar{p}_{\bar{y}}$, namely,

$$\tfrac{1}{2}[\bar{p}_{\bar{y}}(0^+, 0, \bar{z}, \bar{t}) + \bar{p}_{\bar{y}}(0^-, 0, \bar{z}, \bar{t})] = \bar{p}_{\bar{y}}(0, 0^+, \bar{z}, \bar{t}). \tag{3.15}$$

The proof can be carried out in the same manner as that for the two-dimensional case (Ting & Ludloff 1952). A simple proof will be given here by splitting the solution $\bar{p}$, and hence $\bar{p}_{\bar{y}}$, into even and odd solutions of $\bar{x}$. For the even solution there is no discontinuity in $\bar{p}_{e,\bar{y}}$ across the z-axis in the $\bar{x}$–$\bar{z}$ plane. The limit of $\bar{p}_{e,\bar{y}}$ as $\bar{x} \to 0$ is unique and is equal to the value on the left side of (3.15). The odd solution, $\bar{p}_{\mathrm{odd}}$, vanishes on the plane $\bar{x} = 0$, therefore the $\bar{y}$ derivative of $\bar{p}_{\mathrm{odd}}$ vanishes on the plane $\bar{x} = 0$ for $y > 0$. From the sum of the even and odd solution, (3.15) is verified.

With the aid of (3.15) and (3.9) condition (3.10) is replaced by the following condition,

$$\bar{p}_{\bar{y}}(0^+, 0, \bar{z}, \bar{t}) = \rho C^2 (2\mu - A_0 + A_5) f_{x_0 x_0}[\bar{a}\bar{t}, \bar{z}], \tag{3.16}$$

and (3.14) becomes

$$\bar{p}_{\bar{y}\bar{x}}(0^+, 0, \bar{z}, \bar{t}) = \rho C^2 \bar{a}[4M(A_0 - \mu) + A_5\bar{\lambda}_5 + A_0\bar{\lambda}_0] f_{x_0x_0x_0}[\bar{a}\bar{t}, \bar{z}]. \tag{3.17}$$

The differential operator $\bar{D}_{\bar{x}\bar{t}}$ which is the same as that in the two-dimensional problem (Ting & Ludloff 1956) is hyperbolic and can be written as

$$(\partial/\partial\bar{x} + \bar{\lambda}_1 \partial/\partial\bar{t})(\partial/\partial\bar{x} + \bar{\lambda}_2 \partial/\partial\bar{t}).$$

The unknown $\bar{p}_{\bar{y}}(\bar{x} > 0, 0, \bar{t})$, which satisfies the differential equation (3.13) and the boundary conditions (3.16) and (3.17), is obtained in the same manner as the two-dimensional problem (Ting & Ludloff 1952). It takes the form,

$$\bar{p}_{\bar{y}}(\bar{x} > 0, 0, \bar{z}, \bar{t}) = \rho C^2 \sum_{j=1,2,3,4} A_j f_{x_0x_0}[\bar{a}(\bar{t} - \bar{\lambda}_j\bar{x}), \bar{z}]. \tag{3.18}$$

The constants A_j and $\bar{\lambda}_j$ are defined in § 4.

4. The final solution and definitions of symbols

The disturbance pressure behind the shock is given by

$$\begin{aligned} p'(\bar{x}, \bar{y}, \bar{z}, \bar{t}) = \bar{p} + p^* = &-\frac{\rho C^2}{2\pi}\int_{-\infty}^{\infty} d\zeta \int_{-\infty}^{0} d\xi \{A_0 f_{\xi_0\xi_0}(\xi_0, \zeta)/\bar{r} + A_5 f_{\xi_5\xi_5}(\xi_5, \zeta)/\bar{r}\} \\ &-\frac{\rho C^2}{2\pi}\int_{-\infty}^{\infty} d\zeta \int_{0}^{\infty} d\xi \sum_{j=1,2,3,4} A_j f_{\xi_j\xi_j}(\xi_j, \zeta)/\bar{r} \\ &-\frac{\rho C^2 A^*}{2\pi}\int_{-\infty}^{\infty} d\zeta \int_{-\infty}^{\infty} d\xi f_{\xi^*\xi^*}(\xi^*, \zeta)/\bar{r}, \end{aligned} \tag{4.1}$$

where

$$\begin{aligned} \bar{r} &= [(\bar{x} - \xi)^2 + \bar{y}^2 + (\bar{z} - \zeta)^2]^{\frac{1}{2}}, \\ \xi_i &= \bar{a}(\bar{\lambda}_i \xi + \bar{t} - \bar{r}) \quad (i = 1 \text{ or } 5), \\ \xi_j &= \bar{a}(-\bar{\lambda}_j \xi + \bar{t} - \bar{r}) \quad (j = 1, 2, 3, 4), \\ A_0 &= -(\bar{M}_1 - \bar{M}_0 - M)^2, \quad M = U/C, \quad \bar{M}_0 = U_0/C, \quad \bar{M}_1 = U_1/C, \\ \bar{a} &= (\bar{M}_1 + \bar{M}_0)/(1 - M^2)^{\frac{1}{2}}, \quad \bar{\lambda}_0 = M + (1 - M^2)/(\bar{M}_0 + \bar{M}_1), \\ A_1 &= -A_0 H(-\bar{\lambda}_0)/H(\bar{\lambda}_0), \quad \bar{\lambda}_1 = \bar{\lambda}_0. \end{aligned}$$

$\bar{\lambda}_2$, $\bar{\lambda}_3$ are two roots of the quadratic equation,

$$\begin{aligned} H(\bar{\lambda}) &= \bar{\lambda}^2 - 2M\bar{\lambda} + M_0^{-2} = 0, \quad M_0 = U_0/C_0, \\ -A_5 &= A^* = \bar{K}(M/\bar{M}_0)\bar{M}_1^2(1 - M^2)/[(\bar{M}_1 + \bar{M}_0)^2 H(-\bar{\lambda}_5)], \\ \lambda_5 &= \{1 - (M_1^2 - 1)(1 - M^2)/(\bar{M}_0 + \bar{M}_1)^2\}^{\frac{1}{2}}, \\ \lambda_4 &= \lambda_5, \quad A_4 = -A_5 H(-\bar{\lambda}_5)/H(\bar{\lambda}_5), \\ \xi^* &= \bar{a}(\bar{\lambda}^* \xi + \bar{t} - \bar{r}), \quad M_1 = U_1/C_0, \end{aligned}$$

and A_2 and A_3 are the solution of the two linear simultaneous equations,

$$\begin{aligned} A_2 + A_3 &= 2\mu - A_0 - A_1 - A_4 + A_5, \\ \bar{\lambda}_2 A_2 + \bar{\lambda}_3 A_3 &= 4M(\mu - A_0) - A_0\bar{\lambda}_0 - A_1\bar{\lambda}_1 - A_4\bar{\lambda}_4 - A_5\bar{\lambda}_5. \end{aligned}$$

$\bar{K}$ and μ are defined by the equations following (2.26).

In (4.1), the last integral with coefficient A^* is the disturbance pressure p^*, induced by the equivalent wing to remove the inhomogeneous term in the shock equation induced by the disturbances ahead of the shock. The first integral with coefficient A_0 represents the disturbance pressure induced by the position of the wing behind the shock and the induced inhomogeneous terms on the shock condition are removed by its mirror image in the region ahead of the shock, i.e. the integral with the coefficient A_1 and $\bar{\lambda}_1 = \bar{\lambda}_0$. The integral with the coefficient A_5 $(= -A^*)$ and $\bar{\lambda}_5 = \bar{\lambda}^*$ cancels the normal velocity on the x–z plane induced by the presence of the equivalent wing behind the shock and similarly its induced inhomogeneous term is removed by its mirror image, the integral with coefficient A_4 and $\bar{\lambda}_4 = -\bar{\lambda}_5$. The remaining two integrals with coefficients A_2 and A_3 are induced by the image source distribution ahead of the shock. With $\bar{\lambda}_2$ and $\bar{\lambda}_3$ as the two roots of the characteristic equation, the two integrals fulfil the homogeneous shock condition and the coefficients A_2 and A_3 are chosen so that the final solution fulfils the condition of discontinuity at the intersection of the shock with the wing surface. By the inverse Lorentz transformation, the pressure distribution in physical variables x, y, z, t is obtained.

The density variation is obtained from the differential equation (2.5) and the boundary condition (2.14a),

$$\rho'(x,y,z,t) = (1/C^2)\{p'(x,y,z,t) + \Omega_0 p'(x,y,z,t = x/U) \\ + \Omega_3 p_0'[x_0 = (U_1+U_0)\,x/U, y, z]\}.$$

The shock shape is obtained from (2.14c)

$$\psi(y,z,t) = [\rho(U-U_0)]^{-1}\,\Omega_2 \int_0^t p'(x = U\tau, y, z, \tau)\,d\tau \\ + \frac{[1-\Omega_5(M/\bar{M}_0)\,U_1/(U_0-U)]}{(U_1+U_0)}\,\phi_0[x_0 = (U_0+U_1)\,t, y, z],$$

where ϕ is the disturbance velocity potential ahead of the shock with $u_0' = \phi_{0,x_0}$.

5. Reduction to quasi-steady integrals

In the seven integrals in (4.1) the integration variables ξ and ζ are involved implicitly in the first argument of the steady source distribution function. In order to expedite the integration, the variable ξ will be replaced by the first argument of the source distribution function. After this transformation of variables, the last integral becomes a steady three-dimensional solution of an equivalent wing as shown in the appendix. The other six integrals will be reduced to quasi-steady integrals, i.e. the variable $\bar{t}$ appears explicitly in the limit of integration only. From the domains of integration for these new integrals, the boundaries of the disturbed regions behind the shock can be defined directly from the planform of the wing. The limits of integration for ξ for the first two integrals in (4.1) are $-\infty$ and 0 and for the next four integrals are 0 and ∞. For these two groups of integrals the transformation of variables will be discussed separately.

With $\xi = \bar{a}(\bar{\lambda}_i \xi + \bar{t} - \bar{r}),\, i = 0$ or 5, the first group of integrals becomes

$$-\frac{\rho C^2 A_i}{2\pi}\left\{\int_{-\infty}^{\infty} d\zeta \int_{-\infty}^{\tilde{\xi}} F(x_i, \tilde{M}_i, \xi_i, \zeta, \bar{y}, \bar{z})\, d\xi_i \right.$$
$$\left. + \sigma_i \int_{z-\hat{\xi}_i}^{z+\hat{\xi}_i} d\zeta \int_{\tilde{\xi}}^{\xi_{i,m}} F(x_i, \tilde{M}_i, \xi_i, \zeta, \bar{y}, \bar{z})\, d\xi_i \right\}, \tag{5.1}$$

where
$$x_i = \bar{a}(\bar{\lambda}_i \bar{x} + \bar{t}),$$
$$\tilde{M}_i^2 = 1 + \bar{a}^2(1 - \bar{\lambda}_i^2), \quad \tilde{M}_i \gtrless 1, \quad \text{for} \quad \bar{\lambda}_i \lessgtr 1, \tag{5.2}$$
$$F(x_i, \tilde{M}_i, \xi_i, \zeta, \bar{y}, \bar{z}) = f_{\xi_i \xi_i}(\xi_i, \zeta)\{(x_i - \xi_i)^2 - (\tilde{M}_i^2 - 1)[\bar{y}^2 + (\bar{z} - \zeta)^2]\}^{-\frac{1}{2}}, \tag{5.3}$$
$$\tilde{\xi} = \xi_i(\text{at } \xi = 0) = \bar{a}\{\bar{t} - [\bar{x}^2 + \bar{y}^2 + (\bar{z} - \zeta)^2]^{\frac{1}{2}}\}, \tag{5.4}$$
$$\sigma_i = 0, \quad \text{for} \quad \tilde{M}_i < 1, \tag{5.5a}$$
$$\sigma_i = 0, \quad \text{for} \quad \tilde{M}_i > 1 \quad \text{and} \quad \partial\xi_i/\partial\xi \geqslant 0 \quad \text{at} \quad \xi = 0, \tag{5.5b}$$
$$\sigma_i = 2, \quad \text{for} \quad \tilde{M}_i > 1 \quad \text{and} \quad \partial\xi_i/\partial\xi < 0 \quad \text{at} \quad \xi = 0, \tag{5.5c}$$
$$\xi_{i,m} = x_i - \{(M_i^2 - 1)[\bar{y}^2 + (\bar{z} - \zeta)^2]\}^{\frac{1}{2}}, \tag{5.6}$$
$$\hat{\xi}_i = \{(1 - \bar{\lambda}_i^2)\bar{x}^2 - \bar{\lambda}_i^2 \bar{y}^2\}^{\frac{1}{2}}/\bar{\lambda}_i. \tag{5.7}$$

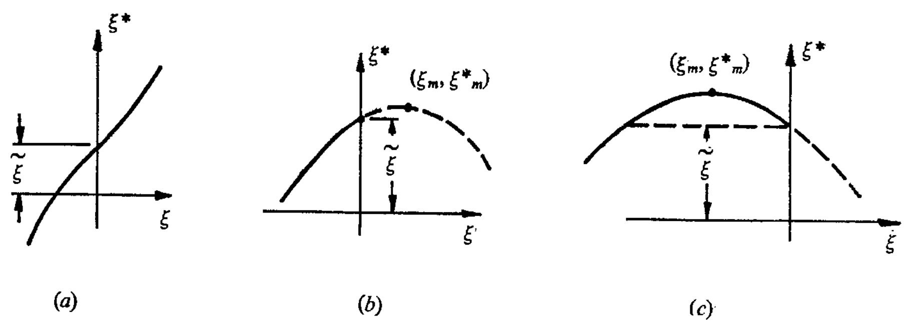


FIGURE 2. Transformation from ξ to ξ^* or ξ_j: (*a*) $1/\bar{\lambda} < 1$, (*b*) $1/\bar{\lambda} > 1$, $\xi_m > 0$, (*c*) $1/\bar{\lambda} > 1$, $\xi_m < 0$.

Intermediate steps in the transformation are supplied in the first part of the appendix with the aid of figure 2. The second integral appears only when conditions in (5.5*c*) are fulfilled. The condition $\partial\xi_i/\partial\xi < 0$ at $\xi = 0$ implies

$$\bar{\lambda}_i + \bar{x}/[\bar{x}^2 + \bar{y}^2 + (\bar{z} - \zeta)^2]^{\frac{1}{2}} < 0. \tag{5.8}$$

This is impossible if $\bar{\lambda}_i < 1$, i.e. $\tilde{M}_i > 1$ and

$$0 < -\bar{\lambda}_i[\bar{y}^2 + (\bar{z} - \zeta)^2]^{\frac{1}{2}}/(1 - \bar{\lambda}_i)^{\frac{1}{2}} < \bar{x}. \tag{5.9}$$

Condition (5.9) in turn defines the limits $\bar{z} \pm \hat{\xi}_i$ for ζ in the second integral.

For the second group of integrals, the variable ξ is replaced by ξ_j with

$$\xi_j = \bar{a}[\bar{t} - \bar{r} - \bar{\lambda}_j \xi] \quad (j = 1, 2, 3, 4),$$

the second group becomes

$$-\frac{\rho C^2 A_j}{2\pi}\int_{-\infty}^{\infty} d\zeta \int_{-\infty}^{\tilde{\xi}} F(x_j, \tilde{M}_j, \xi_j, \zeta, \bar{y}, \bar{z})\, d\xi_j,$$

where $$x_j = \bar{a}(\bar{t}-\bar{\lambda}_j\bar{x}), \tag{5.10}$$

and $$\tilde{M}_j^2 = 1+\bar{a}^2(1-\bar{\lambda}_j^2). \tag{5.11}$$

F and $\tilde{\xi}$ are defined by (5.3) and (5.4). The integrals are evaluated for $\bar{x} < 0$; it is clear that $\partial\xi_j/\partial\xi = \bar{a}[(\bar{x}-\xi)/\bar{r}-\bar{\lambda}_j] < 0$ for $0 \leqslant \xi < \infty$. The negative sign assigned to the square root of the integrand is then cancelled by interchanging the limits. After the transformation of variables, (4.1) becomes

$$\left.\begin{aligned} p'(\bar{x},\bar{y},\bar{z},\bar{t}) &= \bar{p}_J+\bar{p}_0+\bar{p}_5+p^*,\\ \bar{p}_J &= -\frac{\rho C^2}{2\pi}\iint_{\tilde{\Gamma}} d\zeta\, d\xi \sum_{j=0}^{5} A_j F(x_j, \tilde{M}_j, \xi, \zeta, \bar{y}, \bar{z}),\\ \bar{p}_0+\bar{p}_5 &= -\frac{\rho C^2}{2\pi}\sum_{i=0,5}\sigma_i \iint_{\Gamma_i} d\zeta\, d\xi\, A_i F(x_i, \tilde{M}_i, \xi, \zeta, \bar{y}, \bar{z}),\\ p^* &= -\frac{\rho C^2}{\nu\pi}\iint_{\Gamma^*} A_* F(x^*, M_1, \xi, \zeta, \bar{y}, \bar{z})\, d\zeta\, d\xi. \end{aligned}\right\} \tag{5.12}$$

Again F is defined by (5.3). The domain of integration for the first group of integrals is the domain $\tilde{\Gamma}$ inside the hyperbola (figure 3),

$$\tilde{H}: \xi = \bar{a}\{\bar{t}-[\bar{x}^2+\bar{y}^2+(\bar{z}-\zeta)^2]^{\frac{1}{2}}\}. \tag{5.13}$$

The domain of integration for the second group, which is real under condition (5.5*c*), is the domain Γ_i bounded by the hyperbola $\tilde{H}$ and the hyperbola,

$$H_i: \xi = x_i - \{(\tilde{M}_i^2-1)[\bar{y}^2+(\bar{z}-\zeta)^2]\}^{\frac{1}{2}}, \quad \text{for} \quad i = 0, 5. \tag{5.14}$$

H_i and $\tilde{H}$ are tangential to each other at $\zeta_i = \bar{z} \pm \tilde{\zeta}_i$. For the last integral, the domain of integration, Γ^*, is the entire ξ–ζ plane for $M_1 < 1$ and for $M_1 > 1$ is the domain inside the hyperbola,

$$H^*: \xi = x^* - \{(M_1^2-1)[\bar{y}^2+(\bar{z}-\zeta)^2]\}^{\frac{1}{2}}. \tag{5.15}$$

From the definitions of $\tilde{M}_j$, x_j and $\bar{\lambda}_j$, the following relevant results are obtained:

(i) $\tilde{M}_0 = \tilde{M}_1 = (U_0+U_1-U)/C$ = Mach number of wing relative to the undisturbed stream behind the shock.

(ii) $\tilde{M}_2 > 1$, $\tilde{M}_3 > 1$, since $\bar{\lambda}_2 < 1$, $\bar{\lambda}_3 < 1$ and they depend only on the strength of the shock, M_0.

(iii) $M_1 = \tilde{M}_4 = \tilde{M}_5$ = Mach number of the wing relative to the undisturbed stream ahead of the shock.

(iv) x_0 is a co-ordinate fixed on the wing and x_5 and x^* are the same co-ordinates fixed on the fictitious wing.

(v) For $M_1 > 1$, the hyperbolas H_5 and H^* are the same but the domain Γ_5 is contained inside Γ^*.

The domains of integrations in (5.12) and the constant σ_i and ν depend on whether the Mach numbers $\tilde{M}_0$ and M_1 are greater or less than unity, i.e. they depend on the Mach number of the wing relative to the flow behind and to the flow ahead of the shock, respectively. The following are the three possible combinations for $\tilde{M}_0$ and M_1:

(i) $M_1 < 1$, $\tilde{M}_0 < 1$; (ii) $M_1 < 1$, $\tilde{M}_0 > 1$ and (iii) $M_1 > 1$, $\tilde{M}_0 > 1$.

The fourth combination, $M_1 > 1$ and $\tilde{M}_0 < 1$, as shown by the following inequalities and identities, does not exist:

$$[(C_0+U_0-U)^2-C^2]/C_0^2 = 2[(2-\gamma)M_0^2+2M_0-1][M_0^2-1]/[(\gamma+1)M_0^2] > 0, \quad (5.16)$$

and
$$\tilde{M}_0-1 = (M_1C_0+U_0-U-C)/C > (C_0+U_0-U-C)/C > 0. \quad (5.17)$$

For a given wing, the strength of the source distribution vanishes outside the planform S of the wing. The domains of integration for the integrals in (5.12) can therefore be reduced from the appropriate Γ's to their intersection with S (figure 3).

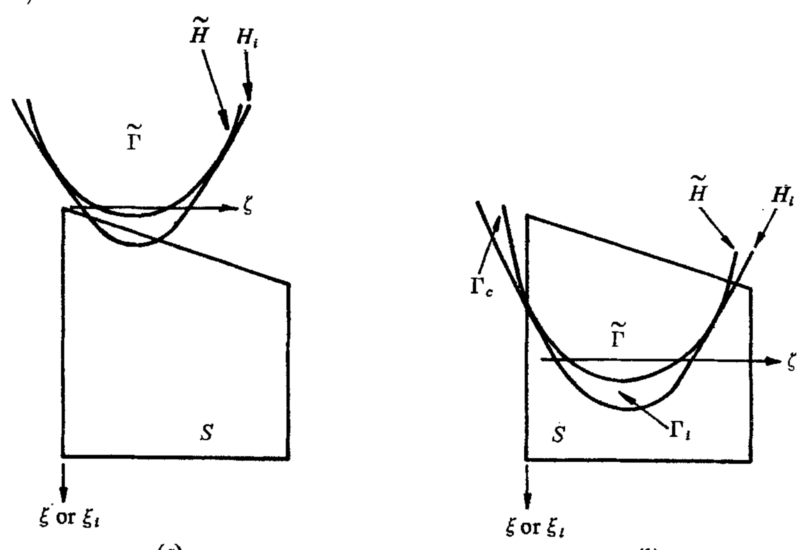


FIGURE 3. Hyperbolas $\tilde{H}$ and H_i (a) for a point in G_2, i.e. outside the domain of influence of the shock, (b) for a point in G_1, i.e. in the domain of influence of the shock.

Let $G_1(\bar{t})$ designate the region in the half space $\bar{x} \leqslant 0$, such that for any point $\bar{x}$, $\bar{y}$, $\bar{z}$ in $G_1(\bar{t})$ the domain of integration for the first group of integrals at the instant $\bar{t}$ is not zero, i.e. $S \cap \tilde{\Gamma} \neq 0$. $S \cap \tilde{\Gamma}$ denotes the intersection of the planform S with the domain $\tilde{\Gamma}$ inside the hyperbola $\tilde{H}$. For points in $G_1(\bar{t})$, the first group of integral p'_J which involves $\bar{t}$ explicitly will not vanish. Furthermore, the image of source distributions due to the shock condition are contained only in the first group of integrals, therefore, domain $G_1(\bar{t})$ will be called the domain of influence of the shock.

Let $G_2(\bar{t})$ designate the complement of $G_1(\bar{t})$ in the half space $x \leqslant 0$, i.e. for any point $\bar{x}, \bar{y}, \bar{z}$ in G_2, $S \cap \tilde{\Gamma} = 0$ and the first group of integrals vanishes, $\bar{p}_J = 0$. The remaining integrals depend on the combinations of M_1 and $\tilde{M}_0$.

(i) For $\tilde{M}_0 < 1$ and $\tilde{M}_5 = M_1 < 1$, $\sigma_0 = \sigma_5 = 0$, the group of integrals $\bar{p}_0$ and $\bar{p}_5$ vanishes. The disturbance pressure p' is given by the steady subsonic solution p^* of the fictitious equivalent wing, i.e.

$$p'(\bar{x},\bar{y},\bar{z},\bar{t}) = p^*(x^*,\bar{y},\bar{z}), \quad \text{for} \quad \bar{x},\bar{y},\bar{z} \quad \text{in} \quad G_2(\bar{t}). \quad (5.18)$$

(ii) For $\tilde{M}_0 > 1$ and $\tilde{M}_5 = M_1 < 1$, $\sigma_0 = 2$ and $\sigma_5 = 0$, the first integral of the second group, $\bar{p}_0$, does not vanish. Since $S \cap \tilde{\Gamma} = 0$, the boundary of the domain of integration $\Gamma_0 \cap S$ will be composed of the boundary of the planform S and the hyperbola H_0. Neither of them depends on t explicitly. The integral $\bar{p}_0$ will therefore be a steady solution in x_0, y, z variables.

(iii) For $\tilde{M}_0 > 1$ and $M_1 > 1$, $\sigma_0 = \sigma_5 = 2$, both integrals $\bar{p}_0$ and $\bar{p}_5$ in the second group do not vanish. The integral $\bar{p}_0$ will have the same properties as that in the proceeding case. Similarly, $\bar{p}_5$ will be a steady solution in x_5, y, z variables.

The domain G_2 can be further subdivided with regions where $\bar{p}_0$ vanishes or $\bar{p}_5$ cancels p^*. The subdivisions for a general planform are presented in Gunzburger (1969). The basic principle for the subdivisions is illustrated in § 6 for a simple planform with straight edges.

6. Examples

The theoretical results will be applied to a wing with a basic planform and thickness distribution as shown in figure 4. The leading edge is $x_0 = kz$ and the two sides are $z = 0$ and $z = B$. The inclination of the upper surface is ϵ, i.e.

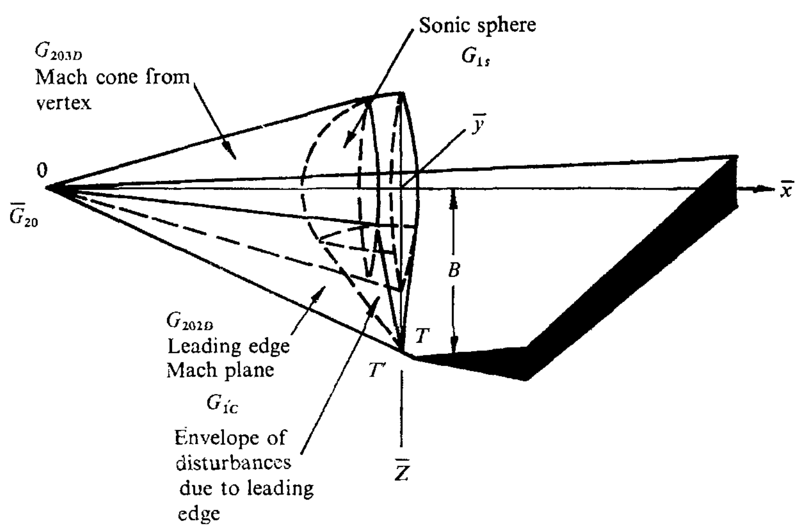


FIGURE 4. Illustration of different regions for wing which is subsonic ahead of shock and supersonic with supersonic edges behind shock.

$f_{x_0}(x_0, z) = 1$. The given planform has two corners, the vertex 0 at the origin of x_0–z axes and the wing tip, T at (kB, B). By superposition of the analytical solutions for this basic wing, numerical results for the pressure distributions and aerodynamic forces for more complicated wings are obtained and presented in § 6 (iii). In order to simplify the description for the various regions of § 5 for the basic planform, it is assumed that at the instant under investigation, the wing tip is still ahead of the shock, i.e. $kB > (U_0 + U_1)\, t$. The span B will not appear in the solution except in $p^*(x^*, y, z)$ when $M_1 < 1$. With the planform of the wing behind the shock having only one corner at 0, the boundaries of various regions will depend on the swept back k and the Mach numbers M_1 and $\tilde{M}_0$.

(i) *Definition of the regions*

The definition of various regions depend on the combinations of supersonic or subsonic Mach numbers M_1 and $\tilde{M}_0$ relative to the stream ahead and behind the shock and the swept back slope k of the leading edge with respect to the Mach cone and the sonic sphere. A special combination is described in detail in this section. Descriptions for all the other possible combinations can be found in Chow & Gunzburger (1969) and Gunzburger (1969).

For the wing moving with subsonic speed relative to the stream ahead of and supersonic to that behind the shock ($M_1 < 1$, $\tilde{M}_0 > 1$) and with a supersonic leading edge $k < \bar{a}(1-\lambda_0^2)^{\frac{1}{2}}$, the region of influence of the shock G_1 is composed of the hemi-sonic sphere G_{1s} and the half cone, G_{1c} with vertex at the intersection of the leading edge with the shock $T'(0, 0, \bar{a}\bar{t}/k)$ and tangential to the sphere. The cone is the envelope of the sonic spheres created by the passing of the leading edge through the shock. The region G_2, which is outside G_1 and behind the shock, can be subdivided to G_{20} and its complement G_{2c}. In G_{2c} the disturbance pressure is p^* alone, induced by the subsonic disturbance created ahead of the shock. In G_{20}, it is the sum of p^* and $\bar{p}_0$, the steady solution for the wing alone.

The boundary between G_{20} and G_{2c} is the Mach cone from the vertex of the wing and the Mach plane from the leading edge. The region G_{20} is composed of two subregions $G_{20,3D}$ and $G_{20,2D}$. $G_{20,3D}$ is bounded by the Mach cone from the vertex 0, the sonic sphere and the half cone containing G_{1c}. For the wing with a constant inclined surface, $\bar{p}_0$ in $G_{20,3D}$ is the same as the steady conical solution and $\bar{p}_0$ in $G_{20,2D}$ is given by the constant value on a wedge with supersonic swept back.

(ii) *Evaluation of integrals and numerical results*

With $f_{x_0x_0}(x_0, z) = 0$ inside the planform, $f_{x_0x_0}$ becomes a δ-function and the double integrals in (5.12) can be carried out immediately and the line integrals with respect to ζ can be written as

$$E(X, \tilde{M}, y, z, \zeta) = \int^{\zeta} d\zeta\{(X-k\zeta)^2-(\tilde{M}^2-1)[y^2+(z-\zeta)^2]\}^{-\frac{1}{2}},$$

$$E = \frac{1}{(\tilde{k})^{\frac{1}{2}}}\sinh^{-1}\frac{\zeta\tilde{k}-Z}{[(1-\tilde{M}^2)I]^{\frac{1}{2}}}, \quad \text{for} \quad \tilde{M} < 1,$$

$$= \frac{-1}{\tilde{k}}\log(X-k\zeta), \quad \text{for} \quad \tilde{M} = 1,$$

$$= \frac{1}{(\tilde{k})^{\frac{1}{2}}}\cosh^{-1}\frac{Z-\zeta\tilde{k}}{[(\tilde{M}^2-1)I]^{\frac{1}{2}}}, \quad \text{for} \quad \tilde{M} > 1, \quad \tilde{k} > 0,$$

$$= -[k^2(y^2+z^2)+X^2-2\zeta Z]^{\frac{1}{2}}/Z, \quad \tilde{M} > 1, \quad \tilde{k} = 0,$$

$$= \frac{1}{(-\tilde{k})^{\frac{1}{2}}}\sin^{-1}\frac{Z-\zeta\tilde{k}}{[(\tilde{M}^2-1)I]^{\frac{1}{2}}}, \quad \tilde{M} > 1, \quad \tilde{k} < 0,$$

where

$$\tilde{k} = k^2+1-\tilde{M}^2, \quad Z = kX+(1+\tilde{M}^2)z \quad \text{and} \quad I = [(X-kz)^2+\tilde{k}y^2]^{\frac{1}{2}}.$$

For the last integral in (5.10) for p', $p^*(x^*, y, z)$ becomes

$$p^* = -A^*[E(x^*, y, z, M_1, S) - E(x^*, y, z, M_1, 0)]/(2\pi), \quad \text{for} \quad M_1 < 1,$$
$$= -A^*[E(x^*, y, z, M_1, \zeta^+) - E(x^*, y, z, M_1, \zeta^-)]/\pi, \quad \text{for} \quad M_1 > 1.$$

ζ^+ and ζ^- are the two roots of the equation, $(x^* - k\zeta)^2 - (M_1^2 - 1)[y^2 + (z-\zeta)^2] = 0$ with $\zeta^+ \geqslant \bar{\zeta}$. If $\zeta^- < 0$, ζ^- is set equal to zero, and if $\zeta^+ < 0, p^*$ is set equal to zero.

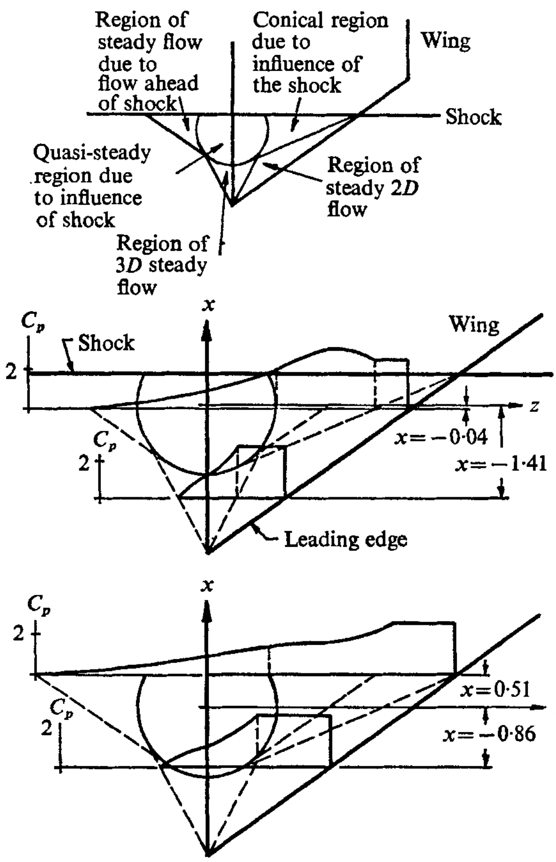


FIGURE 5. Pressure coefficient for supersonic wing with supersonic edges ($M = 0{\cdot}51$, $M_1 = 1{\cdot}5$, $k = 0{\cdot}75$, $y = 0$, $ct = 1$).

For the first group of integrals, $\bar{p}_J$ is equal to zero in region G_2; inside region G_1,

$$\bar{p}_J = -\frac{1}{2\pi} \sum_{j=1,2,3,4,5} A_j[E(x_j, y, z, \tilde{M}_j, \tilde{\zeta}^+) - E(x_j, y, z, \tilde{M}_j, \tilde{\zeta})],$$

where $\tilde{\zeta}^+$ and $\tilde{\zeta}^-$ with $\tilde{\zeta}^+ \geqslant \tilde{\zeta}^-$ are the ζ co-ordinates of the points where the leading edge $\xi = k\zeta$ intersect the hyperbola $\tilde{H}$. If $\tilde{\zeta}^- < 0$, it is replaced by zero.

The second group of integrals $\bar{p}_0$ and $\bar{p}_5$ are non-zero only when the conditions stated in (5.4c) are fulfilled and then they are defined as follows:

$$\bar{p}_i = -A_i[E(x_i, y, z, \tilde{M}_i, \zeta_i^+) - E(x_i, y, z, \tilde{M}_1, \zeta_i^-)]/\pi \quad \text{in} \quad G_2,$$

and

$$\bar{p}_i = -A_i[E(x_i, y, z, \tilde{M}_i, \zeta_i^+) - E(x_i, y, z, \tilde{M}_i, \bar{\zeta}^+) + E(x_i, y, z, \tilde{M}_i, \bar{\zeta}^-) - E(x_i, y, z, \tilde{M}_i, \zeta_i^-)]/\pi \quad \text{in} \quad G_1,$$

where $i = 0$ or 5, ζ_i^+, ζ_i^- are the two roots of

$$(x_i - k\zeta)^2 - (\tilde{M}_i^2 - 1) - [y^2 + (z-\zeta)^2] = 0$$

with $\zeta_i^+ \geqslant \zeta_i^-$. If $\zeta_i^- < 0$, it is replaced by zero.

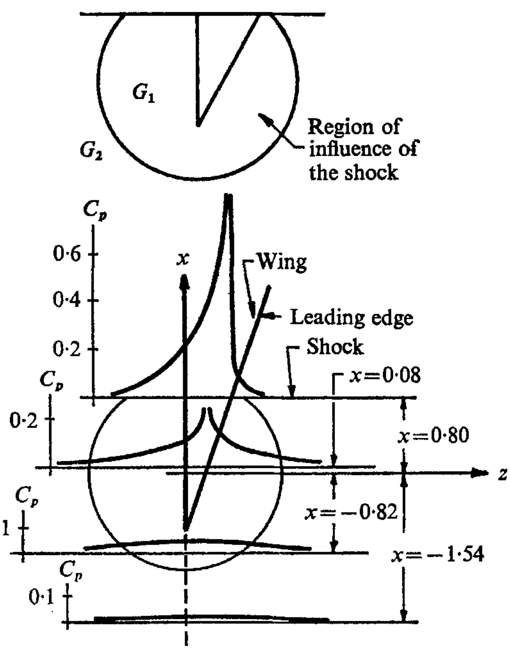


FIGURE 6. Pressure coefficient for a subsonic wing ($M = 0{\cdot}8$, $M_1 = 0{\cdot}25$, $k = 3$, $y = 0$, $ct = 1$).

A numerical program is written to distinguish various regions and to compute from the sum of these explicit expressions for p^*, $\bar{p}_J$, $\bar{p}_0$ and $\bar{p}_5$, the disturbance pressure p' behind the shock. The program yields result for all t, i.e. it works also when the wing tip passes behind the shock. The program can also superpose several basic planforms. Numerical examples for all possible combinations of Mach numbers M_1 and $\tilde{M}_0$ and the swept back k and also for several composite planforms are given in Chow & Gunzburger (1969).

Figures 5 and 6 show two of the numerical examples for the wings with a straight leading edge. The pressure distribution on the wing at various stations of x are shown together with the various domains in x, y, z space. In figure 5 the Mach numbers of the wing with respect to the stream ahead and behind the shock are both supersonic ($M_1 > 1$, $\tilde{M}_0 > 1$).

The characteristics of the pressure distribution in various regions are quite obvious. The discontinuities in the slope of the pressure curve as it crosses the boundaries of various domains, e.g. the sonic sphere, the Mach cone, are quite obvious. In particular, along the intersection of the wing with the shock, $x = MCt$, the disturbance pressure is constant outside the Mach cone of the equivalent wing and is the value along a ray from the vertex T' of the conical solution. At T' the pressure is not single valued. It ranges from the two-dimensional value behind the oblique shock attached to the leading edge and the conical values along the rays from the vertex T' to zero ahead of the leading edge. In figure 6, the wing is moving at subsonic speed ($M_1 < 1$, $\tilde{M}_0 < 1$). Outside the region G_1 the disturbance pressure is the subsonic steady solution $p^*(x^*, y, z)$ corresponding to a wing moving at velocity $(1-\bar{\lambda}^*M)\,C/(\bar{\lambda}^*-M) = U^*$ in x–t variables. The pressure distribution behind the shock for $t < 0$ can be obtained from the present result by a translation of x co-ordinate, e.g. the pressure distribution at the instant $t_0 > 0$ at $x = -2Ct_0$ is the same as that at $x = -(2C-U^*)\,t_0$ at the instant $t = 0$.

(iii) *Applications*

For a thin symmetric wing with an arbitrary planform and thickness distribution, the pressure disturbance behind the shock wave can be obtained directly from (4.1) or (5.12) by numerical evaluation of the double integrals. For wings designed for high-speed flight, the planform can be decomposed to several triangles and the inclination of the surface in each triangle is a constant. The pressure distribution for such wings can be obtained by superposition of the explicit solutions given in § 6 (i) and § 6 (ii) for wings with the basic planform in the same manner as in the steady three-dimensional problems (Donovan & Lawrence 1957).

For a wing at an angle of attack moving at supersonic speed and with supersonic edges relative to the stream behind the shock, the flow fields above and below the wing are not influenced by each other and by the flow field behind the trailing edge. The pressure distribution on the top and the bottom surfaces can therefore be computed by the analysis of this paper for wings with equivalent symmetric thickness distributions.

Figure 7 shows the results of the calculations for a triangular plate at an angle of attack and moving at supersonic speed ($M_1 > 1$, $\bar{M}_1+\bar{M}_0-M > 1$) and with supersonic edges. Before the impinging of the shock by the wing ($t < 0$), the lift and drag on the plate are given by the steady flow solution in the stream behind the shock with Mach number M_1, i.e. (Liepman & Roshko 1957)

$$L_0 = D/\alpha = \rho_0 U_1^2 \bar{C}_{p_0}(X^2/k), \quad \bar{C}_{p_0} = 2\alpha/(M_1^2-1)^{\frac{1}{2}},$$

where L and D are the lift and drag, X is the mid-chord length and $2X/k$ is the span. $\bar{C}_{p_0}$ is the spanwise mean of pressure coefficient, and is equal to the two-dimensional value (Donovan & Lawrence 1957).

When the shock wave intercepts the wing, $X/(U_1+U_0) > t > 0$, the pressure distribution on the wing ahead of shock remains unchanged and that behind the shock is conical, i.e. p' is a function of $x_0/(Ct)$, $y/(Ct)$ and $z/(Ct)$. The lift variation on the wing is

$$L(t) = \rho_0 U_1^2 \bar{C}_{p_0}\{X^2-[(U_1+U_0)\,t]^2\}/k + \rho C^4 \alpha \bar{J}(\bar{M}_1+\bar{M}_0)\,t^2,$$

where
$$J(\tau) = -4\int_0^\tau d\left(\frac{x_0}{ct}\right)\int_0^{x_0/(kCt)} p'\left(\frac{x_0}{Ct}, 0^+, \frac{z}{Ct}\right) d\left(\frac{z}{Ct}\right),$$

$$\bar{M}_1 = U_1/C, \quad \bar{M}_0 = U/C, \quad x_0 = x+(U_0+U_1-U)\,t,$$

and p' is obtained from the explicit solution in § 6 (ii) with a superposition of the latter's mirror image with respect to the x–y plane. The lift curve during this period is therefore a parabola as shown in figure 7.

When the trailing edge has passed through the shock and intercepts the sonic sphere, $X/(U_1+U_0-U-C) > t > X/(U_1+U_0)$, the lift is given by the expression $L(t) = \rho C^4 \alpha J(X/Ct)\, t^2$. The lift curve in this interval is not a parabola as shown in figure 7.

When the trailing edge passes through the sonic sphere, $t > X/(U_1+U_0-U-C)$, the wing is outside the domain of influence of the shock. The lift on the wing is

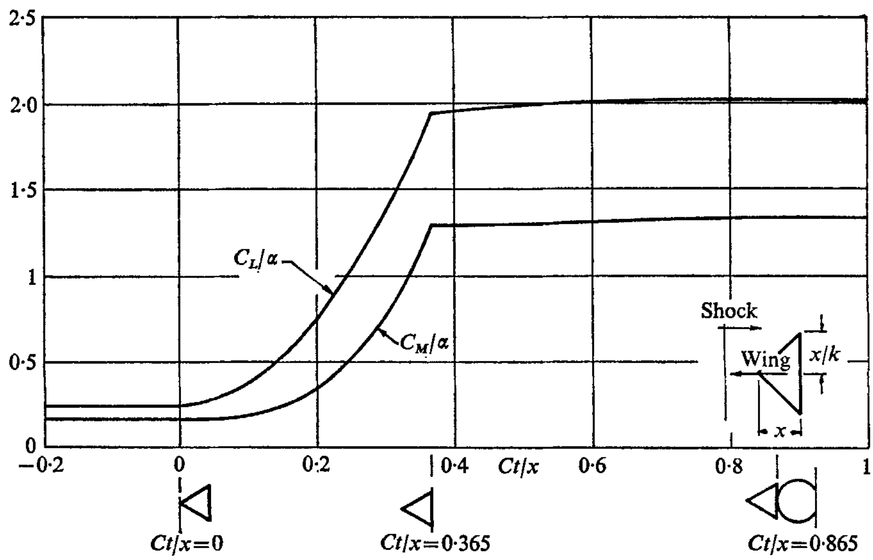


FIGURE 7. Lift and moment coefficients *vs.* non-dimensionalized time for wing and supersonic leading edges ($M = 0{\cdot}51$, $M_1 = 1{\cdot}5$, $k = 0{\cdot}75$). Wing Mach number $= 1{\cdot}5$. Pressure ratio across shock $= 7{\cdot}3$. $C_L = \mathscr{L}k/\tfrac{1}{2}\rho \bar{U}_0^2 x^2$; $C_M = \mathscr{M}k/\tfrac{1}{2}\rho \bar{U}_0^2 x^3$; α = angle of attack.

therefore a constant (figure 7) and is given by the steady supersonic solution with respect to the stream behind the shock with Mach number, $\bar{M}_1+\bar{M}_0-M$. Also shown is the variation of moment about the leading edge.

Figure 8 shows the variation of centre of pressure. It moves forward from the 2/3 chord position in steady flow to about 0·46 and then finally returns to the 2/3 chord position after the trailing edge has passed over the sonic sphere. Figure 9 shows the pressure variation on a flat terrain in the shape of a pyramid when the shock wave has passed over it. The pressure distribution is obtained by superposition of the explicit solution in § 6 (ii) three times corresponding to the three swept back edges with $M_1 = 0$ and their images with respect to the x–y plane. Due to the symmetry with respect to the x–y plane the pressure distribution is shown for half of the pyramid ($z > 0$). The locations of the discontinuities in the slope of the pressure curves which are pre-determined from the boundaries for various regions described before, are quite essential in drawing the pressure curves for computed data points.

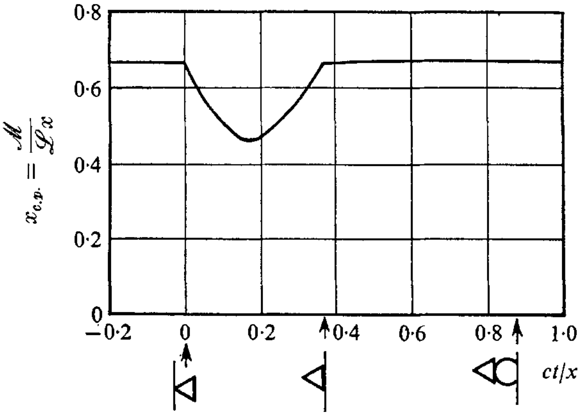


FIGURE 8. Variation of centre of pressure *vs.* non-dimensionalized time for wing in figure 7. Wing Mach number = 1·5. Pressure ratio across shock = 7·3.

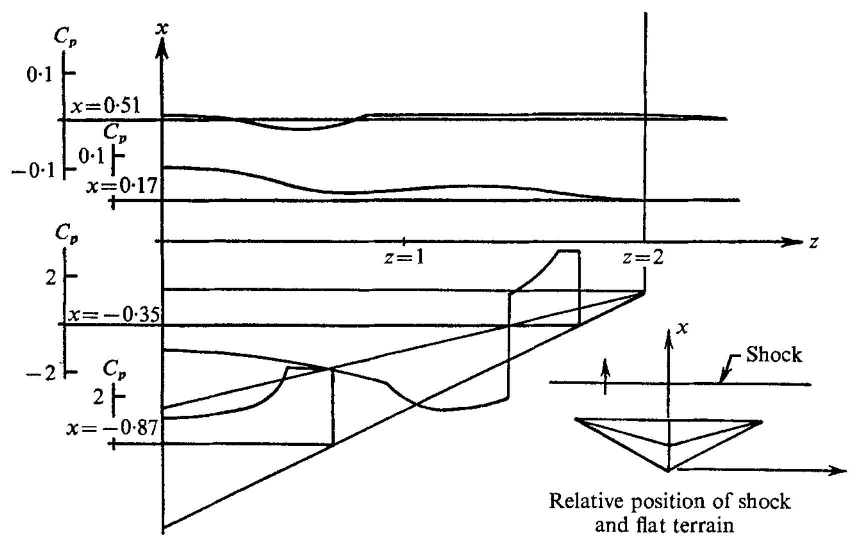


FIGURE 9. Pressure coefficient on thin pyramid like obstacle on the ground after the shock wave has passed over it. (M = 0·51, M_1 = 0, k_1 = 0·5, k_2 = 2·5, k_3 = 0.)

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Appendix. Reduction of an unsteady solution to a steady solution

An unsteady solution of the wave equation (2.6) in the physical variables x, y, z, t with the speed of sound C is also a solution of the wave equation (2.20) in the Lorentz variables $\bar{x}, \bar{y}, \bar{z}, \bar{t}$ with the speed of sound equal to unity. For a planar source distribution moving with uniform speed $1/\bar{\lambda}$, the solution of (2.20) can be written as

$$\bar{\phi}(\bar{x}, \bar{y}, \bar{z}, \bar{t}) = -\iint_{-\infty}^{\infty} g[a(\bar{\lambda}\xi + \bar{t} - \bar{r}), \zeta]\, d\xi\, d\zeta/(2\pi\bar{r}) \tag{A 1}$$

$\bar{\lambda}^{-1}$ is of course also the Mach number of the moving source distribution. In the physical variables, it is moving with Mach number $M_\infty = (1-\bar{\lambda}M)/(\bar{\lambda}-M)$ and velocity $M_\infty C$. Note that $M_\infty - 1$ and $(1/\bar{\lambda})-1$ have the same sign. With x^* as co-ordinate fixed on the source distribution, i.e. $x^* = x + M_\infty Ct$, it is quite obvious that the unsteady solution ϕ with respect to the stream behind the shock should be equivalent to a supersonic or subsonic steady solution $\phi(x^*, y, z)$ with the speed of sound C. A brief derivation of their equivalence will be given in § (i) of the appendix. In § (ii), a steady solution with respect to the stream ahead of the shock which cannot be equivalent to an unsteady solution behind the shock in the whole space, is made equivalent on a special plane, say the plane of shock.

(i) *Equivalence of an unsteady solution to a steady solution*

By replacing the variable ξ by $\xi^* = a(\bar{\lambda}\xi + \bar{t} - \bar{r})$, (A 1) becomes

$$\bar{\phi} = \frac{-1}{2\pi}\int_{-\infty}^{\infty} d\zeta \int \frac{g(\xi^*, \zeta)\, d\xi^*}{[\bar{r}(\partial\xi^*/\partial\xi)]}. \tag{A 2}$$

After expressing $\bar{r}$ in terms of ξ^*, the denominator in the integrand becomes

$$\bar{r}\{\partial\xi^*/\partial\xi\} = \bar{r}\{a[\bar{\lambda} + (\bar{x}-\xi)/\bar{r}]\} = \pm\{(x^*-\xi^*)^2 - (M_\infty^2 - 1)[\bar{y}^2 + (\bar{z}-\zeta)^2]\}^{\frac{1}{2}}, \tag{A 3}$$

with
$$M_\infty^2 - 1 = a^2(1-\bar{\lambda}^2) \quad \text{and} \quad x^* = a(\bar{t} + \bar{\lambda}\bar{x}). \tag{A 4}$$

Both x^* and M_∞ will agree with their physical definitions given before when

$$a = M_\infty/(1-M^2)^{\frac{1}{2}}. \tag{A 5}$$

The choice of the appropriate sign for the denominator and that of the limits of integration in (A 2) should be decided by the sign of $\partial\xi^*/\partial\xi$. $\partial\xi^*/\partial\xi > 0$ for all ξ if $1/\bar{\lambda} < 1$, i.e. the motion of the source distribution is subsonic. ξ^* increases monotonically from $-\infty$ to ∞ as ξ does and (A 2) becomes

$$\bar{\phi} = \phi(x^*, y, z) = \frac{-1}{\pi\nu}\iint_{\Gamma^*} \frac{g(\xi^*, \zeta)\, d\xi^*\, d\zeta}{\{(x^*-\xi^*)^2 + (1-M_\infty^2)[y^2 + (z-\zeta)^2]\}^{\frac{1}{2}}}. \tag{A 6}$$

For $\bar{\lambda}^{-1} < 1$, $\nu = 2$ and Γ^* is the entire ξ^*–ζ plane and $\phi(x^*, y, z)$ represents a steady subsonic solution ($M_\infty < 1$).

For the supersonic case, $1/\bar{\lambda} > 1$, $\partial\xi^*/\partial\xi$ has the same sign as $\bar{x} - \xi - \bar{\lambda}\bar{r}$. They vanish at $\xi = \xi_m$ and $\xi^* = \xi_m^*$, with

$$\xi_m = \bar{x} - \bar{\lambda}\{[\bar{y}^2 + (\bar{z}-\zeta)^2]/(1-\bar{\lambda}^2)\}^{\frac{1}{2}}, \quad \xi_m^* = x^* - \{(M_\infty^2 - 1)[\bar{y}^2 + (\bar{z}-\zeta)^2]\}^{\frac{1}{2}}.$$

As ξ increases from $-\infty$ to ξ_m, ξ^* increases from $-\infty$ to ξ_m^* and the positive sign in (A 3) should be used. As ξ increases from ξ_m to ∞, ξ^* decreases from ξ_m^* to $-\infty$ and the negative sign in (A 3) should be used. For the supersonic case, $\bar{\lambda} < 1$, (A 2) again becomes (A 6) with $\nu = 1$ and Γ^* being the domain inside the hyperbola $x^* - \xi^* = \{(M_\infty^2 - 1)[y^2 + (z-\zeta)^2]\}^{\frac{1}{2}}$ and $\phi(x^*, y, z)$ represents a steady supersonic solution ($M_\infty > 1$). Thus concludes the proof of the equivalence.

(ii) *Matching of a steady solution ahead of the shock on the plane of the shock with an unsteady solution behind the shock*

The inhomogeneous term $\bar{K}p'_{0,x_0x_0}[x_0 = \bar{a}\bar{t}, \bar{y} > 0, \bar{z}]$ in the shock condition (2.24) is associated with a steady solution ahead of the shock defined by (2.4) with the speed of sound C_0. The inhomogeneous term will remain unchanged if the variable x_0 which is fixed on the wing is replaced by a new variable x^* which is related to $\bar{x}, \bar{t}$ by a linear transformation $x^* = \bar{a}(\bar{t}+\bar{\lambda}^*\bar{x})$ so that $x^* = x_0 = \bar{a}\bar{t}$ on the plane of the shock $\bar{x} = 0$. Elsewhere x^* and x_0 are not the same. The constant λ^* is free to be defined. The condition of (3.5) that $D_{\bar{x}\bar{t}}p^*$ at $\bar{x} = 0$ matches with the inhomogeneous term $\bar{K}p'_{0,x_0,x_0}(x_0 = \bar{a}\bar{t}, \bar{y}, \bar{z})$ will be fulfilled if $D_{\bar{x}\bar{t}}p^*(\bar{x}, \bar{y}, \bar{z}, \bar{t})$ is identified with $\bar{K}p'_{0,x^*x^*}(x^*, \bar{y}, \bar{z})$, i.e.

$$-\frac{\rho C^2}{2\pi}\bar{a}^2A^*H(\bar{\lambda}^*)\iint_{-\infty}^{\infty}\frac{d\xi\,d\zeta}{\bar{r}}f^{(\mathrm{IV})}[\bar{a}(\bar{\lambda}^*\xi+\bar{t}-\bar{r}),\zeta]$$
$$= \frac{\rho_0U_1^2\bar{K}}{\pi\nu}\iint_{\Gamma^*}\frac{f^{(\mathrm{IV})}(\xi^*,\zeta)\,d\xi^*\,d\zeta}{\{(x^*-\xi^*)^2-(M_1^2-1)\,[(\bar{z}-\zeta)^2+\bar{y}^2]\}^{\frac{1}{2}}}, \quad \text{(A 7)}$$

where $f^{(\mathrm{IV})}$ means the fourth derivative with respect to the first argument. By comparison with (A 1) and (A 6), the constants A^* and $\bar{\lambda}^*$ are defined,

$$\bar{\lambda}^* = \{1-(M_1^2-1)\,(1-M^2)/(\bar{M}_1+\bar{M}_0)^2\}^{\frac{1}{2}}$$

and
$$A^* = -(M/\bar{M}_0)\,\bar{M}_1^2\bar{K}(1-M^2)/[(\bar{M}_1+\bar{M}_0)^2H(-\bar{\lambda}^*)]. \quad \text{(A 8)}$$

When x^* is related to $\bar{x}, \bar{t}$ and in turn to the physical variables x, t, $A^*f(x^*, z)$ can be considered as a fictitious source distribution for an equivalent wing. From (A 8), it is clear that the equivalent wing moves with supersonic speed, $1/\bar{\lambda}^* > 1$ (or at subsonic speed, $1/\bar{\lambda}^* < 1$) with respect to the stream behind the shock when the original wing is moving with respect to the stream ahead of the shock at supersonic speed, $M_1 > 1$ (or subsonic speed $M_1 < 1$).

For the subsonic case $\bar{\lambda}^*$ given by (A 8) is always real. For the supersonic case, $M_1 > 1$, $\bar{\lambda}^*$ given by (A 8) can be imaginary or zero for certain combinations of M_1 and M_0. This possibility will be investigated.

For a supersonic flow ahead of the shock, the radius of intersection of the shock and the Mach cone is $R_0 = (\bar{M}_1+\bar{M}_0)\,\bar{t}/[(M_1^2-1)\,(1-M^2)]^{\frac{1}{2}}$. The radius of the intersection of the shock and the Mach cone of the equivalent wing moving with Mach number $1/\bar{\lambda}^*$ in $\bar{x}, \bar{t}$ variables is $R^* = \bar{t}/(1-\bar{\lambda}^{*2})$. A necessary condition for the equivalence of those two solutions in the plane of the shock is that $R_0 = R^*$. R^* has a lower bound $\bar{t}$ which is the radius of the sonic circle. When the radius R_0 is less than $\bar{t}$, $\bar{\lambda}^*$ is imaginary. This means only that the solution cannot be represented by the type of (3.1). It does not mean that the mathematical problem stated at the end of § 2 has no solution.

The condition for $\bar{\lambda}^*$ real is $R_0 > \bar{t}$ which is equivalent to the condition,

$$[(\gamma-1)\,(M_0^2-1)^2-(3-\gamma)]\,M_1 < (\gamma+1)\,M_0^3$$
$$+\{[2(\gamma-1)\,M_0^2+4]\,(M_0^2-1)\,[M_0^4+(\gamma-1)\,M_0^2+1]\}^{\frac{1}{2}}. \quad \text{(A 9)}$$

The inequality holds for all values of M_1 if $M_0^2 < 1+[(3-\gamma)/(\gamma-1)]^{\frac{1}{2}}$, i.e. $M_0 < 3$ for $\gamma = 1\cdot4$ or the shock strength $p/p_0 < 3\cdot7$. For stronger shock the inequality defines an upper bound for M_1, e.g. with $p/p_0 = 20$, $M_0 = 4{\cdot}16$, $M_1 < 4{\cdot}50$. It is clear that so long as the shock strength is less than 20, the proposed procedure for the removal of the inhomogeneous term works for wings moving at super- or subsonic speeds. Indeed, in the solution by transform method of the problem for a supersonic moving wing the same restriction was imposed by a statement in Arora (1968) which amounts to assuming $(\bar{\lambda}^*)^2 > 0$.

REFERENCES

ARORA, N. L. 1968 An integral-transform method for shock–shock interaction studies. *J. Fluid Mech.* **34**, 209.

ARORA, N. L. 1969 Integral transforms for shock–shock interaction—three dimensional planar wings. *J. Appl. Math. Phys.* **20**, 244.

BAKER, E. B. & COPSON, E. T. 1950 *The Mathematical Theory of Huygen's Principle.* Oxford University Press.

BLANKENSHIP, V. D. 1965 Shock–shock interaction on a slender supersonic cone. *J. Fluid Mech.* **22**, 599.

CHESTER, W. 1954 The diffraction and reflection of shock waves. *Quart. J. Mech. Appl. Math.* **7**, 57.

CHOW, F. & GUNZBURGER, M. 1969 Numerical program and examples for diffraction of shock by wings. *New York University Report* NYU-AA-69-8.

DONOVAN, A. F. & LAWRENCE, H. R. 1957 *Aerodynamic Components of Aircraft at High Speeds.* Section A. Princeton, N.J.: Princeton University Press.

GUNZBURGER, M. 1969 Diffraction of shock wave by a thin wing-symmetric and anti-symmetric problem. Ph.D. Thesis, School of Engineering and Science, New York University.

LIEPMAN, H. W. & ROSHKO, A. 1957 *Elements of Gasdynamics.* New York: John Wiley.

LIGHTHILL, M. J. 1949 The diffraction of blast I. *Proc. Roy. Soc.* A **198**, 454.

LUDLOFF, H. F. & FRIEDMAN, M. B. 1952 Diffraction of blasts by axisymmetric bodies. *J. Aero. Sci.* **19**, 425.

SMYRL, J. F. 1963 The impact of a shock wave on a thin two-dimensional aerofoil moving at supersonic speed. *J. Fluid Mech.* **15**, 223.

TER-MINASSIANTS, S. M. 1969 The diffraction accompanying the regular reflexion of a plane obliquely impinging shock wave from the walls of an obtuse wedge. *J. Fluid Mech.* **35**, 391.

TING, L. & LUDLOFF, H. F. 1952 Aerodynamics of blasts. *J. Aero. Sci.* **19**, 317.